ON THE CHOI-EFFROS MULTIPLICATION

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ABSTRACT. A short proof is given for the well-known Choi-Effros theorem on the structure of ranges of completely positive projections.

In this note an alternate proof is given for the following well-known and basic theorem of M. D. Choi and E. G. Effros (see Theorem 3.1 in [1]):

Theorem 1. *Let A be a C^*-algebra and let $\Phi : A \to A$ be a completely positive, contractive and idempotent linear map. Then there exist a C^*-algebra B and a complete order isomorphism $\rho : B \to Ran(\Phi)$ such that $\rho(ab) = \Phi(\rho(a)\rho(b))$ for all $a, b \in B$.*

Proof. We may assume that A is generated, as a C^*-algebra, by $Ran(\Phi)$. Let J denote the closed right ideal of A generated by all operators of the form $xy - \Phi(xy)$ with $x, y \in Ran(\Phi)$. We will show that $Ker(\Phi) = J$. Let $z \in Ker(\Phi)$ with $z \geq 0$ and let $y \in A$. Then, by the Kadison-Schwarz inequality,

$$\Phi(zy)\Phi(zy)^* \leq \|z^{1/2}y\|^2\Phi(z) = 0.$$

In particular this holds true when $z = \Phi(x^*x) - x^*x$ for some $x \in Ran(\Phi)$. This shows that $J \subset Ker(\Phi)$. We will now show, by induction over k, that if $u = x_1 \cdots x_k$ with $x_j \in Ran(\Phi)$ for $j = 1, \ldots, k$ then $u - \Phi(u) \in J$. When $k = 1$ or $k = 2$ this is obvious. Suppose that $k \geq 3$ and assume it holds for $k - 1$. Write $u = u_1 + u_2$ where

$$u_1 = (x_1x_2 - \Phi(x_1x_2))x_3 \cdots x_k$$

and

$$u_2 = \Phi(x_1x_2)x_3 \cdots x_k.$$

Then $u_1 \in J$ hence $\Phi(u_1) = 0$, and $u_2 - \Phi(u_2) \in J$ by the induction hypothesis. It then follows that $Ker(\Phi) = J$ and, in particular, that $Ket(\Phi)$ is a bilateral ideal in A. Let $B = A/Ker(\Phi)$ and let $\rho : B \to Ran(\Phi)$ be the quotient map induced by Φ. It is now routine to see that the couple (B, ρ) has all the properties we were looking for.

□

REFERENCES

[1] M.D. Choi, E.G. Effros, *Injectivity and operator spaces*, J. Functional Analysis , **24** (1977), no. 2, 156–209.

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